Mathematical Preliminaries

1 Sets

- A set is an **unordered** collection of distinct objects (*elements*). These elements may be anything, including other sets.
- Special sets include:
  - N - set of natural numbers (integers beginning with 0).
  - R or $\mathbb{R}$ - set of real numbers.
  - $\{\}$ - null or empty set.
- The elements of a set may be specified in a number of ways, the simplest being:
  $$V = \{a, e, u, o, i\}.$$
  The notation $a \in V$ means "a is an element of V."

  Another way to specify a set is to describe the elements logically in "set-builder notation":
  $$S = \{x | x \in \mathbb{R} \text{ and } 0 \leq x \leq 1\}$$
  is the set of all real numbers in the interval from 0 to 1 inclusive.
- S is a *subset* of T if and only if (*iff*) each element of S is also an element of T, and is denoted by $S \subseteq T$.
- The *power set* of S, denoted by $P(S)$, is the collection of all subsets of S.
- The *cardinality* of a set S, denoted by $|S|$, is the number of distinct elements of S.
- New sets may be formed from existing sets S and T:
  - *union*: $S \cup T$.
  - *intersection*: $S \cap T$.
  - *difference*: $S - T$.
  - *complement*: $\overline{S}$.
- Two sets, A and B, are *disjoint* if $A \cap B = \{\}$.

2 Functions

- A function f is a mapping from one set, A, called the *domain*, to another set B, called the *co-domain*, that associates **each** element of A with a **single** element of B:

  If $a \in A$ and f maps a to b, we write $f(a) = b$.

- The *range* of f is the set containing all $b \in B$ such that $f(a) = b$ for some $a \in A$.

- Special functions of interest:

  - Floor: $f(x) = \lfloor x \rfloor$. This is the largest integer less than or equal to x.
  - Ceiling: $f(x) = \lceil x \rceil$. This is the smallest integer greater than or equal to x.
  - Factorial: $f(n) = n!$. Note that the domain of this function is the natural numbers.

- Exponentiation (base b): $f(x) = b^x$. Rules for manipulation:

$$b^x b^y = b^{x+y} \tag{1}$$
$$\frac{b^x}{b^y} = b^{x-y} \tag{2}$$
$$(b^x)^y = b^{xy} \neq b^{(x^y)} \tag{3}$$

- Logarithm ($b > 0$): $f(x) = \log_b x$. This means $y = \log_b x$ if $b^y = x$. If b is unspecified, **2 is assumed** in this course.

  - Some common notation:

$$\lg n = \log_2 n \tag{4}$$
$$\lg n^2 = \log_2 n^2 = \log_2(n^2) \tag{5}$$
$$\lg^2 n = \log_2^2 n = (\log_2 n)^2 \tag{6}$$

  - Rules for manipulation:

$$\log_b 1 = 0 \tag{7}$$
$$\log_b b = 1 \tag{8}$$
$$\log_a x = \frac{\log_b x}{\log_b a} \quad \text{where} \quad b > 0 \tag{9}$$
$$\log xy = \log x + \log y \tag{10}$$
$$\log x^y = y \log x \neq (\log x)^y \tag{11}$$
$$\log x < x \quad \text{for all } x > 0 \tag{12}$$
$$\frac{d}{dx} \log_b x = \frac{1}{x \ln b} \tag{13}$$

3 Summations

- A summation is a short-hand notation to describe the addition of the terms of a sequence:

$$\sum_{i=m}^{n} f(i) = f(m) + f(m+1) + f(m+2) + ... + f(n)$$

  The variable i is called the index variable. Usually $m = 0$ or $m = 1$.

- Special summation formulas:

$$\sum_{i=0}^{n} 2^i = 2^{n+1} - 1 \quad (14)$$

$$\sum_{i=0}^{n} a^i = \frac{a^{n+1} - 1}{a - 1} \quad \text{where} \quad a \neq 1 \quad (15)$$

$$\sum_{i=0|1}^{n} i = \frac{n(n+1)}{2} \quad (16)$$

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \quad (17)$$

$$\sum_{i=1}^{n} \frac{1}{i} \approx \log_e n \quad (18)$$

- Some rules for manipulating summations, where c is a constant:

$$\sum_{i=1}^{n} ca_i = c\sum_{i=1}^{n} a_i \quad (19)$$

$$\sum_{i=m}^{n} (a_i \pm b_i) = \sum_{i=m}^{n} a_i \pm \sum_{i=m}^{n} b_i \quad (20)$$

$$\sum_{i=m}^{n} c = c(n - m + 1) \quad (21)$$

$$\sum_{i=m}^{n} a_i = \sum_{i=0}^{n} a_i - \sum_{i=0}^{m-1} a_i \quad (22)$$

$$\sum_{i=m}^{n} a_i = \sum_{j=0}^{n-m} a(j + m) \quad (23)$$